一般化スペクトル理論とその応用

千葉 逸人 [1]

[1] 東北大学 材料科学高等研究所　〒 980-8577 仙台市青葉区片平 2-1-1

あらまし　行列に対する固有値の概念を無限次元空間上へと一般化したスペクトル理論は古くより知られているが，ここでは，連続スペクトルや剰余スペクトルを扱うためにそれをさらに一般化した理論，およびその様々な問題への応用を紹介する．
キーワード 一般化スペクトル理論，ゲルファンドの 3 組，同期現象

The generalized spectral theory and its applications

Hayato CHIBA[1]

[1] Tohoku University, Advanced Institute for Materials Research (AIMR)
2-1-1 Katahira, Aoba-ku, Sendai, 980-8577, Japan

Abstract The spectral theory of linear operators is well studied as a generalization of eigenvalues of matrices. In this article, the spectral theory is further generalized to treat a continuous and residual spectrum, and several applications will be given.
Key words Generalized spectral theory, Gelfand triplet, Synchronization

1. Introduction

An eigenvalue of a matrix is a fundamental tool in any area of mathematics. For example, for a linear differential equation $dx/dt = Ax,\ x \in \mathbf{C}^n$ defined on a finite dimensional space, eigenvalues provide the stability of solutions.

When a given space is an infinite dimensional space $\mathcal{H}$, the concept of the set of eigenvalues is extended to the spectrum set, which consists of not only usual eigenvalues but also the continuous spectrum and residual spectrum. Because of the continuous spectrum and residual spectrum, the analysis of a linear differential equation $dx/dt = Ax,\ x \in \mathcal{H}$ becomes too difficult.

To overcome this difficulty, a generalized spectral theory based on a Gelfand triplet is developed [1]. It is applied to

- the stability of the steady state of a linear differential equation [1].
- the analysis of the Kuramoto model, in particular the stability and bifurcation of the synchronized state [2].
- behavior of Schrödinger equations [3].
- the existence of a stable brain wave [5].
- chaotic behavior of symbolic dynamical systems [4].

- application to the estimation of the computing performance of the reservoir computing (to appear).
- the mechanism to become diabetes (in progress).

etc... The purpose of this article is to give a brief review of the generalized spectral theory. Please refer to [1] more precise and detailed results.

2. Gelfand triplet

Let consider the linear equation $dx/dt = Tx$, $x \in \mathcal{H}$ defined on an infinite dimensional Hilbert space $\mathcal{H}$. It is known that a solution is given by the Laplace inversion formula

$$x(t) = e^{Tt}x(0) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\lambda t}(\lambda - T)^{-1}x(0)d\lambda, \tag{1}$$

for $t > 0$, where the integral path is a vertical straight line such that the spectrum set of T is included in the left half plane $\mathrm{Re}(\lambda) < a$; Fig. 1 (left). The operator e^{Tt} is called the semigroup generated by T. The set of singularities of the integrand $(\lambda - T)^{-1}$ is the spectrum set. If it consists only of discrete eigenvalues of T, we can calculate the Laplace inversion formula by deforming the integral path and using the residue theorem as is shown in Fig. 1 (right). Hence, the real parts of eigenvalues completely determine the asymptotic behavior of solutions because of the factor $e^{\lambda t}$.

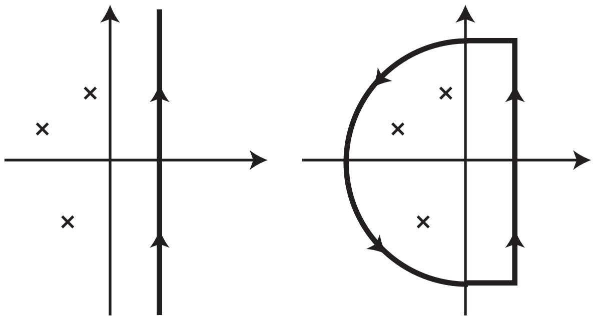

図 1: A deformation of the integral path. $\times$ denotes an eigenvalue.

Suppose that T has a continuous spectrum on the imaginary axis. The Kuramoto model is this case. In this case, we can not deform the path from the right to the left half plane because imaginary axis itself is the set of singularity. Thus, it is difficult to investigate the stability of solution in the usual Hilbert space theory.

To handle the difficulty caused by the continuous spectrum on the imaginary axis, we develop the generalized spectral theory based on a Gelfand triplet. In this section, we will illustrate how the triplet naturally arises by a simple example.

Let us consider the multiplication operator $\mathcal{M} : f(x) \mapsto xf(x)$ on $L^2(\mathbf{R})$. The continuous spectrum is the whole real axis. Indeed, the resolvent is given by

$$(\lambda - \mathcal{M})^{-1}f(x) = \frac{1}{\lambda - x}f(x)$$

and it is not included in $L^2(\mathbf{R})$ when $\lambda \in \mathbf{R}$. Nevertheless, we will show that there exists a topological vector space larger than $L^2(\mathbf{R})$ on which the resolvent operator makes sense even if $\lambda \in \mathbf{R}$.

To this end, we consider the $L^2(\mathbf{R})$-inner product with some functions ϕ, ψ

$$((\lambda - \mathcal{M})^{-1}\phi, \psi^*) = \int_{\mathbf{R}} \frac{1}{\lambda - x}\phi(x)\psi(x)dx,$$

where $\psi^*(x) := \overline{\psi(\overline{x})}$ is introduced to avoid the complex conjugate in the right hand side. The right hand side above is holomorphic in λ on the lower half plane $\{\mathrm{Im}(\lambda) < 0\}$.

Next, suppose λ approaches the real axis from below

$$\lim_{\mathrm{Im}(\lambda)\to 0} \int_{\mathbf{R}} \frac{1}{\lambda - x} \phi(x)\psi(x)dx.$$

The factor $1/(\lambda - x)$ diverges at $x = \lambda \in \mathbf{R}$, however, it is known that as long as ϕ and ψ are continuous functions on $\mathbf{R}$, the above integral exists as an improper integral and is continuous in $\lambda \in \mathbf{R}$.

Further suppose that λ moves to the upper half plane. It is known that as long as ϕ and ψ are holomorphic on the region $\{\mathrm{Im}(\lambda) \geq 0\}$, the above function of λ has an analytic continuation to the upper half plane given by

$$\int_{\mathbf{R}} \frac{1}{\lambda - x} \phi(x)\psi(x)dx + 2\pi i \phi(\lambda)\psi(\lambda), \quad \mathrm{Im}(\lambda) > 0.$$

Now we have shown that if ϕ and ψ are holomorphic on the real axis and the upper half plane, the function $((\lambda - \mathcal{M})^{-1}\phi, \psi^*)$ of λ has an analytic continuation from the lower to the upper half plane across the continuous spectrum on the real axis. We denote it as

$$R(\lambda; \phi, \psi) := \begin{cases} \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - x} \phi(x)\psi(x)dx, & \mathrm{Im}(\lambda) < 0 \\ \displaystyle\int_{\mathbf{R}} \frac{1}{\lambda - x} \phi(x)\psi(x)dx + 2\pi i \phi(\lambda)\psi(\lambda), & \mathrm{Im}(\lambda) > 0. \end{cases}$$

Motivated by this observation, let X be a dense subspace of $L^2(\mathbf{R})$ consisting of some class of holomorphic functions and X' be its dual space, the vector space of continuous linear functionals on X. The mapping $\phi \mapsto R(\lambda; \phi, \psi)$ defines a linear functional on X, which is denoted by $R(\lambda; \bullet, \psi) \in X'$. The topology on X is defined so that this functional is continuous. Then, the mapping $\psi \mapsto R(\lambda; \bullet, \psi)$ gives a linear mapping from X to X', denoted by $\mathcal{R}_\lambda$, that is holomorphic in $\lambda \in \mathbf{C}$. By the definition, $\mathcal{R}_\lambda = (\lambda - \mathcal{M})^{-1}$ when $\mathrm{Im}(\lambda) < 0$. We call $\mathcal{R}_\lambda$ the generalized resolvent of $\mathcal{M}$.

This discussion is summarized as follows: As an operator from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$, the resolvent operator $(\lambda - \mathcal{M})^{-1}$ is singular on the real axis because of the continuous spectrum. Nevertheless, if we regard it as an operator from X into X', it has an analytic continuation $\mathcal{R}_\lambda$ from the lower to the upper half plane. For any $\psi \in X$, $\mathcal{R}_\lambda \psi$ is an X'-valued holomorphic function.

If X is a dense subspace of $L^2(\mathbf{R})$ and the embedding is continuous, $L^2(\mathbf{R})$ is continuously embedded to the dual space X'. In this manner, we obtain the triplet

$$X \subset L^2(\mathbf{R}) \subset X' \tag{2}$$

called the **Gelfand triplet** or **rigged Hilbert space**.

3. Generalized spectrum

The spectrum set is also generalized as follows. Let $\mathcal{H}$ be a Hilbert space and T a linear operator on $\mathcal{H}$. Recall that the spectrum set of T is the set of singularities of the resolvent $(\lambda - T)^{-1}$. Suppose that T has a continuous spectrum. In a similar manner to the above, suppose that there exists a suitable subspace $X \subset \mathcal{H}$ such that if we regard the resolvent as an operator from X to X', then it has an analytic continuation $\mathcal{R}_\lambda$ across the continuous spectrum. In general, the Riemann surface of $\mathcal{R}_\lambda$ is nontrivial. If the analytic continuation $\mathcal{R}_\lambda$ has a new singularity on the Riemann surface different from the original complex plane, we call it a **generalized spectrum**. By the definition, it is not a true eigenvalue in $\mathcal{H}$-sense, however, it is expected that it plays a similar role to a usual eigenvalue and provides a new information that is not obtained from the framework of a Hilbert space.

Recall that the semigroup e^{Tt} generated by T is given by the Laplace inversion formula (1). Again suppose that T has a continuous spectrum on the imaginary axis and we cannot deform the integral path from the right to the left half plane. Now we assume that there exists a subspace $X \subset \mathcal{H}$ such that the resolvent $(\lambda - T)^{-1}$ has an analytic continuation $\mathcal{R}_\lambda$ from the right to the left half plane as an operator from X into X'. Hence, we interpret (1) as

$$e^{Tt}\phi = \lim_{y\to\infty} \frac{1}{2\pi i}\int_{a-iy}^{a+iy} e^{\lambda t}\mathcal{R}_\lambda \phi \, d\lambda, \quad \phi \in X. \tag{3}$$

Then, we can deform the integral path toward the left half plane (more precisely, the second sheet of the Riemann surface), on which $\mathcal{R}_\lambda \phi \in X'$ [1]. A singularity of $\mathcal{R}_\lambda$ on the second Riemann sheet is called the generalized eigenvalue. By picking up the residue of the generalized eigenvalue, we can estimate the asymptotic behavior of the semigroup.

4. Application to the Kuramoto model

It is applied to the dynamics of the Kuramoto model as follows. At first, we give a brief review of the Kuramoto model.

The **Kuramoto model** is one of the most famous coupled oscillators given by

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N, \tag{4}$$

which is well-known as a typical mathematical model for synchronization phenomena [6, 7]. Here, ω_i and K are constants called natural frequencies and the coupling strength, respectively. When the coupling strength is zero, there are no interactions between oscillators and they rotate with their own velocity ω_i. Hence, if $\omega_j > \omega_i$ then θ_j overtakes θ_i many times. However, if K is positive, there are interactions between oscillators through the term $\sin(\theta_j - \theta_i)$ and we expect that if K is large enough, such an overtaking does not occur. Indeed, it is easy to observe by numerics that there exists a threshold K_c such that when $K > K_c$, a synchronized state appears; a subset of oscillators forms a cluster on a circle and it behaves like a big oscillator without overtaking. The cluster consists of oscillators whose natural frequency ω_i is close to the average Ω of all natural frequencies. As K increases, the number of oscillators that are entrained into the cluster gets larger.

図 2: (left) synchronization. (right) de-synchronization.

In order to observe that whether a synchronization occurs or not, it is convenient to introduce the **order parameter** defined by

$$\eta := \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)}. \tag{5}$$

[1]Thus, the limit $\lim_{y\to\infty}$ in (3) is considered in weak sense (weak dual topology on X').

This gives the center of mass of oscillators on a unit circle. Hence, when its absolute value $r := |\eta|$ is positive (resp. zero), a synchronization occurs (resp. does not occur). Kuramoto performed a certain formal and technical calculation using the order parameter, and reached the following result, though there are no mathematical proofs.

The Kuramoto conjecture [6, 7].

Suppose $N \to \infty$ and the natural frequencies are independent and identically distributed according to a probability density function $g(\omega)$. If $g(\omega)$ is an even and unimodal function, a bifurcation diagram of the order parameter $r = |\eta|$ is given as Fig. 3. This means that when K is smaller than $K_c := 2/(\pi g(0))$, the de-synchronized state $r = 0$ is asymptotically stable. At $K = K_c$, a bifurcation (phase transition) occurs and a stable synchronized state ($r > 0$) exists for $K > K_c$. Near the bifurcation point, r is approximately given by $r \sim O(\sqrt{K - K_c})$.

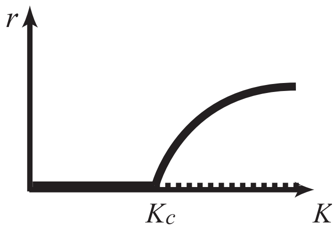

図 3: A bifurcation diagram of the order parameter.

The bifurcation point $K_c := 2/(\pi g(0))$ is often called Kuramoto's transition point. See [6] for Kuramoto's formal calculation.

The difficulty of a mathematical approach to the Kuramoto conjecture is that a certain linear operator obtained by the linearization of the model has a continuous spectrum on the imaginary axis. Hence, we employ the generalized spectrum theory.

(1) When $K > K_c$, the spectrum consists of continuous spectrum and unique eigenvalue. The de-synchro state $r = 0$ is unstable because the eigenvalue lies on the right half plane; Fig. 4 (left).
(2) As $K \to K_c$, the eigenvalue goes to the left side and at the bifurcation point $K = K_c$, it is absorbed into the continuous spectrum and disappears, in the usual Hilbert space theory.
(3) Actually, the eigenvalue does not disappear. Even if $K < K_c$, if we apply the generalized spectrum theory, we can show that it still exists on the second Riemann sheet as a generalized eigenvalue; Fig. 4 (right). Since the (generalized) eigenvalue across the imaginary axis at $K = K_c$, we can apply the bifurcation theory (center manifold theory in generalized sense).

In this manner, the Kuramoto conjecture was proved [2].

In the following theorems, $h(\theta)$ denotes a distribution of the initial values $\{\theta_j(0)\}_{j=1}^{\infty}$ of oscillators.

Theorem 1.

Suppose that $g(\omega)$ is the Gaussian distribution. When $0 < K < K_c$, there exists $\delta > 0$ such that if $h(\theta)$ satisfies

$$\left| \int_0^{2\pi} e^{ij\theta} h(\theta) d\theta \right| < \delta, \quad j = 1, 2, \cdots,$$

then the order parameter $\eta(t)$ tends to zero as $t \to \infty$ with an exponential rate.

Theorem 2. Suppose that $g(\omega)$ is the Gaussian distribution. There exist numbers $\varepsilon_0, \delta > 0$ such that if $h(\theta)$

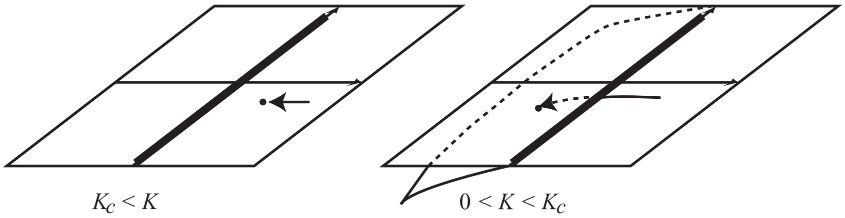

図 4: The motion of the (generalized) eigenvalue as K decreases. When $K > K_c$, it is a usual eigenvalue in L^2-sense. When $0 < K < K_c$, it is a generalized eigenvalue that lies on the second Riemann sheet different from the original complex plane.

satisfies

$$\left| \int_0^{2\pi} e^{ij\theta} h(\theta) d\theta \right| < \delta, \quad j = 1, 2, \cdots,$$

then for $K_c < K < K_c + \varepsilon_0$, the absolute value of the order parameter converges to the following value as $t \to \infty$

$$|\eta(t)| = \sqrt{\frac{-16}{\pi K_c^4 g''(0)}} \sqrt{K - K_c} + O(K - K_c).$$

In particular, a bifurcation diagram of the order parameter is given as Fig. 3.

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